Examining liquidity risk and calculating value at risk of portfolio taking into account the liquidity aspect of equities

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Abstract

This dissertation was written as part of the EMBA at the International Hellenic University. The recent crisis has highlighted the importance of liquidity for the correct functioning of financial markets and the banking sector. Many financial institutions faced problems because they did not manage their liquidity more circumspectly. They depended too much on short-term sources of funding and on funding from other institutions and at the same time they did not hold sufficient stocks of liquid assets to be in a position to handle a deterioration in funding conditions. The crisis in liquidity in many funding markets in 2007-08 demonstrated how quickly liquidity can disappear and proved that the institutions have to change their approach on managing liquidity risk.

Having experienced these events, the global financial system is entering a new era in which liquidity will be a key element. During the last few years, value at risk has become a favored tool for measuring market risk across financial institutions. However, the classical VaR modeling ignores the presence of a liquidity component. This component arises from the hypothesis that the theoretical selling off implied by the VaR calculations takes place at the mid-price. The main goal of this dissertation is to investigate the aspect of the liquidity of a portfolio of assets and more specifically of stocks. In the first part we present liquidity and associated risk along with the main measures of liquidity. In the second part we introduce the Value at Risk concept of a portfolio and how to incorporate liquidity in its calculation. Different models are presented. The third part is the empirical analysis. The collected time series data were used to calculate VaR and LVaR with different methods and for different portfolios to illustrate how liquidity can increase the calculated value at risk of a portfolio.

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Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>III</td>
</tr>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>III</td>
</tr>
<tr>
<td>CONTENTS</td>
<td>IV</td>
</tr>
<tr>
<td>INTRODUCTION TO LIQUIDITY RISK</td>
<td>6</td>
</tr>
<tr>
<td>LIQUIDITY</td>
<td>7</td>
</tr>
<tr>
<td>LIQUIDITY MEASURES</td>
<td>8</td>
</tr>
<tr>
<td>THE VaR CONCEPT</td>
<td>10</td>
</tr>
<tr>
<td>Delta normal method</td>
<td>11</td>
</tr>
<tr>
<td>Historical simulation method</td>
<td>11</td>
</tr>
<tr>
<td>Monte Carlo method</td>
<td>11</td>
</tr>
<tr>
<td>THE LVaR CONCEPT</td>
<td>12</td>
</tr>
<tr>
<td>EMPIRICAL ANALYSIS</td>
<td>17</td>
</tr>
<tr>
<td>Data</td>
<td>18</td>
</tr>
<tr>
<td>Historical volatility measures</td>
<td>18</td>
</tr>
<tr>
<td>Bid-ask spread estimator</td>
<td>19</td>
</tr>
<tr>
<td>Numerical calculations for a single stock portfolio</td>
<td>21</td>
</tr>
<tr>
<td>Numerical calculations for multiple stocks portfolio</td>
<td>27</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>32</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>33</td>
</tr>
</tbody>
</table>
Introduction to Liquidity risk

Liquidity risk is a crucial and confusing subject. All large financial institutions spend a lot of resources every year trying to manage liquidity risk. The term itself covers a wide range of issues.

- **Quality of transactions, assets and markets.**
- **Balance sheet and funding.** Adds the issue of the terms on which credit is granted. Traders and investors want to be able to maintain a leveraged position in case of a decline in their perceived credit quality or of the value of posted collateral. In banking and insurance the question is institution’s solvency.
- **Quality of the financial system as a whole.** In cases of severe financial crisis the ability of the financial system to allocate credit, support markets or settle financial transactions is questioned. This is usually called systemic risk and is a function of economy-wide liquidity.

In our analysis we will focus on the first, since it is more directly related to the transaction cost and thus the cost of entering and exiting portfolio positions. The following chart, from a research paper of Citigroup, shows the impact of a liquidity bubble on price and risk. As Citi observes, “all a liquidity tsunami does for credit, as well as for equity, is to perpetuate the illusion of maximum pricing while shifting the risk curve to the point where any deviation from "perfection" - or loss of faith in the liquidity or its providers will ultimately lead to an instantaneous waterfall in price”.

![Image of chart showing impact of liquidity on price and risk.](source:Citi Research)
Liquidity

Liquidity refers to an individual asset, portfolio or transaction or even to a market segment. An asset is liquid if it is a good substitute for cash, so it can be traded quickly and without generating an effect on its price. When used to describe a market, it refers to the ability to liquefy a portfolio of assets in an orderly fashion with small transaction costs and without affecting the price. The liquidity of a market is important; given liquidity, market participants are able to hedge positions in other fixed income securities, to speculate on interest rates and to correctly price other securities such as derivatives on interest rates.

Liquidity is a multi-dimensional variable and can be examined from different points of view. The five main aspects are the following:

- Trading time is the ability to perform a transaction immediately without affecting the price. Measures for trading time are the waiting time between trades and the number of trades per unit of time.
- Tightness is the ability to perform a round-trip transaction at almost the same price at the same time. Tightness shows the transaction costs and indicates how far transaction costs diverge from mid-market prices. It is measured by the various versions of bid/ask spread and by brokers’ commissions.
- Depth is the maximum volume of trades that can be performed for any given bid/ask spread without affecting the quoted price. Shallowness in the market is a sign of illiquidity which affects the trading investor adversely. Apart from depth itself, order ratio and flow ratio are strong indicators of depth.
- Resiliency is the ability to quickly revert to initial price levels after a large transaction. Resiliency, contrary to depth, also takes into account the elasticity of supply and demand. Measurements of resiliency include intraday returns or the liquidity ratio.
- Immediacy is the speed with which a transaction of a given size can be executed. It incorporates elements of depth and resiliency so it is not clearly a separate dimension. Amihud and Mendelson (1986) showed that wider bid-ask spreads are linked with higher returns. This means that the market adds a premium on securities with greater immediacy. So liquidity is a priced risk factor.

Based on the above aspects we can identify five different levels of liquidity

1) In the first level there is no liquidity in the market, so no transaction can be executed.
2) In the second level it is possible to buy or sell only a certain amount of a security/asset and this will affect the quoted price.
3) In the third level it is also possible buy or sell only a certain amount of a security without affecting the quoted price. The more liquid the market becomes the less the transaction affects the quoted price.
4) In the fourth it is possible to buy and to sell a security at almost the same price and same time.
5) In the fifth level it is possible to execute any transaction immediately and without affecting the price.

Liquidity risk arises from the imperfections of the markets which affect the five above-mentioned aspects. These imperfections are a result of the cost of searching for a counterparty and the financial institutions that assist in the search. They can be divided in four categories.

- **Cost of trade processing.** Includes fixed and variable cost of processing, clearing and settling trades. These costs depend on the state of technology and the organization of the markets themselves. Over reasonably long periods of time they can be considered stable.

- **Inventory management by dealers.** Dealers try to provide market participants with the possibility of making immediate transactions, so they attempt to predict the market-clearing price on the basis of which they will hold long or short inventories of an asset. This exposes them to price risk which must be compensated by price volatility.

- **Adverse selection.** This is related to privileged access to information of some traders who have more possibilities to forecast the equilibrium price. It is extremely difficult for dealers to identify which offers to trade are due to a counterparty’s intention to reposition and which because a counterparty realizes that the prevailing price is wrong.

- **Differences of opinion.** Investors usually have different opinions about the “proper” price of an asset or on how to evaluate a new piece of information. When unexpected information first becomes known or during times of crisis, it is more difficult to find a counterparty.

### Liquidity measures

Liquidity as such cannot be observed and so it has to be approached through different liquidity measures. Traders that need liquidity are usually not patient and they are ready to pay a premium to liquidate their position or build a position quickly. At the same time, there are patient traders willing to supply liquidity to them. Most liquidity measures refer to the cost arising from demanding liquidity or, seen from the other side, the compensation needed to supply liquidity. This section looks at the most conventional measures and their impact on liquidity. Which measure is preferred depends principally on the purpose of the analysis and the available data.

- **Issued amount** is the size of the issuance expressed in money value or as the number of asset units. As McCauley & Remolana note, the trading turnover in cash and futures will generally be higher as the outstanding stock of publicly issued central government debt increases. Higher turnover means greater liquidity. (McCauley Remolona 2000)
- Age refers to the time period from issuance. Its importance lies in the market segmentation by asset maturity (Martinez Resano 2005). More mature securities have less liquidity.

- Missing prices occur when the security was not traded during a given time interval. If the price of a security at the end of one day is the same as that of the previous day it is highly possible the security did not trade that day (Houweling 2003). The more missing prices occur, the less the liquidity.

- Bid-ask spreads:
  
  Quoted bid-ask spread is “the gap between quoted bid and ask prices and is observed before an actual transaction takes place” (CGFS 1999). It represents the cost of executing a specific trade of limited size. In the case of executing large positions over a time horizon of some days, the bid-ask spread fails to capture exactly the liquidity cost. Fleming noted that as a measure to compare different securities, it needs to be adjusted for time horizons (Fleming 2003)

  Realized bid-ask spread is the difference between bid and ask price averages for executed trades over a period of time weighted by the transaction volumes at each price (CGFS 1999).

  Effective bid-ask spread is calculated by doubling the the difference between the transaction price and the mid-quote identified immediately before the transaction (Goldreich 2005) The effective bid-ask spread takes account of the change in the price between quotation and actual transaction (CGFS 1999).

  High bid-ask spreads are a signal of low liquidity.

- Quoted size is the amount of securities explicitly bid for or offered for sale at the posted bid and offer prices. Big quotes give high liquidity. Fleming notes, however, that this measure may underestimate depth as “market makers usually do not reveal the full quantity they want to transact” (Fleming 2003)

- Best liquidity is the average of the quoted size at the best bid and offer taking into consideration the quotes immediately prior to the transactions (Dunne 2006)

  Total liquidity is the average of the total amount offered and the total amount bid in the best three quotes when taking into consideration only the quotes immediately prior to the transactions (Dunne 2006). Both these measures are positively related with liquidity.

- Bid-side market depth is “the difference between bid and mid-price, divided by the bid quantity” (Favero 2005)

  Ask-side market depth is “the difference between ask price and mid-price, divided by the ask quantity” (Favero 2005)

  Both these measures are positively related with liquidity.

- Liquidity premium arises from the difference in security liquidities. This is often taken as represented by the difference between the yields on on-the-run and off-the-run securities of similar cash flow characteristics (Fleming 2003). This measure has both advantages and disadvantages. On the one hand,
calculation does not need high-frequency data. Moreover, the liquidity spread reflects price liquidity as well as differences in security liquidities. However, other factors may also bring about a price premium with respect to on-the-run security trades, a fact which undermines its usefulness as a liquidity measure.

- **Trading volume/trading frequency** refer to the total value of assets traded in a time unit and the number of transactions executed in a specified time interval. Both values are positively related to price volatility, which increases as liquidity falls.

- **Price volatility** is taken to be an indication of differences in liquidity, as it may reflect changes in “bid-ask spread, the market impact of trades and/or the degree of resiliency” (CGFS 1999).

- **Cost of round trip** is the cost (expressed as a percentage) of buying and selling a certain number of shares simultaneously through the submission of market orders. For any specific transaction size, it is the total status of the limit order book at any time. Lower round trip cost is an indicator of greater liquidity (Irvine 2000).

---

**The VaR concept**

Value at risk (VaR) of a portfolio of assets refers to the maximum losses which may be incurred with a given probability over a set time horizon. It is the value that, with a particular probability, will not be exceeded over a specific time horizon. The value of VaR will be exceeded with some frequency.

There are two types of VaR estimations, the relative VaR and the absolute VaR. In the first, losses are defined relatively to expected value and in the latter losses are defined relatively to the initial position. According to Jorion (2001) VaR can be calculated with 3 different methods. These methods differ in their assumptions with respect to risk factor distribution and whether they use linear or full valuation.

If the initial value of a portfolio is \( W_0 \) then its value at the end of the examined period will be \( W = W_0 (1+R) \) with \( R \) representing the returns. \( E(R) = \mu \) and \( V(R)=\sigma^2 \). \( R^* \) is the worst possible return and \( W^* \) the worst possible value of the portfolio under a certain confidence level. Relative and absolute VaR can be calculated by the following formulas

\[
\text{VaR} = E(W) - W^* = W_0 (R^*-\mu) \quad \text{(relative)}
\]
\[
\text{VaR}^* = -W_0 - W^* = -W_0 R^* \quad \text{(absolute)}
\]

The worst portfolio value will not be exceeded with probability \( P(w\leq W^*) = 1-c \), where \( c \) is the confidence level. We assume normal distribution and calculate worst return using standard normal distribution.

\[
P(R < R^*) = P(Z < \frac{R^* - \mu}{\sigma}) = 1 - c,
\]

\( R^* = \mu + \alpha*\sigma \) where \( \alpha<0 \) is quantile of standard normal distribution.

---

1 Assuming a constant fundamental level of prices (CGFS 1999)
\[
\text{VaR} = -\alpha \sigma W_0 \quad \text{and} \quad \text{VaR}' = -(\alpha^\star \sigma + \mu)W_0
\]

**Delta-Normal method (Parametric VaR, No data)** assumes that all asset returns are normally distributed and so the portfolio’s return also follows a normal distribution based on it being a linear combination of normal variables. So taking values over time we compute all the risk factor variances and correlations. The VaR of the portfolio can then be calculated by combining the linear exposures to all risk factors, also assumed to be normally distributed, and from the covariance matrix forecast. The input data required are volatility and correlation forecasts for each risk factor and positions on risk factors. It is the simplest method to implement and easy to scale over periods of time. A drawback of this method is the assumption that all risk factors follow the normal distribution which is not exactly the case in practice.

**Historical-Simulation Method (Nonparametric, Historical data)** consists of going back in time and using time series of historical returns weighted by current positions. As, Jorion notes (1996), if the returns are all normally distributed, the VaR calculated through the historical-simulation method would be the same as that calculated by the delta-normal method. The input data required are a time series of actual movements for each risk factor and current positions on risk factors. This is also relatively easy to implement since nowadays there is a huge amount of historical data available on price changes through internet databases. Distributions do not need to be normal nor securities linear. A disadvantage to the method, however, is that the use of a single sample path may not be sufficient to give a good forecast of future distributions.

**Monte Carlo method (Future data)** is the most sophisticated method and is suitable for any distribution and even for non-linear securities. First we have to specify a stochastic process for financial variables as well as for process parameters. The type of distributions and process parameters such as risk and correlations can be derived from historical data. Next, we simulate future hypothetical price paths for all variables of interest. The time horizon considered can vary from one day to even months ahead. The portfolio is fully valued at mark-to-market. Each fictitious realization is then used to construct a distribution of log returns. VaR can then easily be calculated depending on the chosen confidence level. The input data required include specification of a stochastic process for each risk factor, valuation models for all assets in the portfolio and positions on various securities. It is a method which demands significant computational time and thorough comprehension of the stochastic process employed. In case of stock prices the most suitable distribution is the Geometric Brownian.

The Value at Risk (VaR) measure was developed originally to measure market risk. Market risk occurs due to the fluctuations in level or volatility of market prices so VaR is calculated on the basis of mid-prices. However, if a certain portfolio is liquidated the transactions will not take place with the mid-price because the volume
of the transaction may affect the price. The actual price will depend on the liquidity of the market.

The liquidity risk cannot be ignored because this will lead to the calculation of lower VaR values, resulting in greater real losses. In the following section we will examine ways to incorporate liquidity risk in the VaR model.

**The LVaR concept**

Liquidity adjusted VaR (LVaR) differs from conventional VaR because it takes into consideration the size of the initial holding position and the liquidity impact. Liquidity impact includes exogenous liquidity factors reflected in the bid-ask spread and endogenous liquidity factors witnessed by the price movement brought about by trading.

Models developed to capture endogenous liquidity risk aim to identify optimal liquidation strategies for any position. If the liquidation is immediate the execution costs are high. On the other hand if the liquidation process is slow, exposure to price risk is higher. The optimal trading strategy tries to capture the best balance between execution costs and price risk. Based on the optimal strategy, LVaR can be derived. Models trying to capture exogenous liquidity risk focus the spread distribution. Certain modifications to this type of model also allow the endogenous liquidity risk to be captured.

Shamroukh (2000) suggests a model for calculating LVaR beginning from one asset and one risk factor. It begins with the calculation of mean and variance of portfolio value defined once liquidation is complete. The critical point is that liquidation of the portfolio takes place in parts over the holding period. Thus the liquidation scheme consists of dates and volumes of trading over a period T. Assuming normal distribution for risk factor level, calculation of the model will give the variance of the portfolio value, on the basis of which LVaR can be calculated as for usual VaR. The value differs though from the original VaR because the liquidation is taken as occurring over the holding period. This difference represents the liquidation factor and depends on the number of trading dates. As the number of trading dates tends to infinity, the liquidation factor tends to 1/3.

Almgren & Chriss (1999) developed the idea of LVaR in the frame work of identifying the optimal strategy for portfolio liquidation. It is based on the fact that rationally a trader always tries to minimize the expectation of shortfall for any given level of shortfall variance. The strategy chosen would be the most efficient, with the minimal error in its estimate of expected liquidation cost, that is, the strategy with the lowest variance for the same or a lower level of expected transaction costs. A parametre of all possible efficient strategies will be a single variable representing all possible maximum levels of variance in transaction costs. From this one can arrive at Almgren & Chriss’s “efficient frontier of optimal trading strategies”.

-12-
Assuming that we have a basket of securities $S_1, S_2, \ldots, S_N$ each of them following an arithmetic Brownian motion.

\[ dS_i = \mu_i dt + \sigma_i \, dz_i \]

where the coefficient $\mu_i$ is the drift and the coefficient $\sigma_i$ is the volatility and $dz_i$ is the standard Brownian motion. $\Sigma$ is the covariance matrix relating $dz_i$’s. Since at the time of liquidation there is no particular view of what direction the stock will move we set $\mu_i = 0$. The liquidation will take place from now (time 0) to later (time T), trading over equally spaced periods of time $t_1, t_2, \ldots, t_M$, where $\tau = t_i - t_{i-1}$ is constant. The transaction cost functions can be described in terms of number of shares traded over period $\tau$. The transaction costs have two components, permanent and temporary. “Permanent costs refer to market impact that persists for the life of the liquidation. That is, if we execute a trade and in so doing the market moves its consensus view of the price, then this impact will be felt in subsequent transaction levels. On the other hand, temporary costs refer to purely liquidity based costs that reflect the market’s short horizon premium for providing liquidity. These costs are not reflected in subsequent transaction prices and are thus referred to as temporary”. Assuming linear functions for both of them. The temporary impact expressed as a function of the rate of trading per unit time $\tau$ is:

\[
\text{temporary impact} = c \text{sgn}(n) + \eta \, \frac{n}{\tau}
\]  

(1)

This shows that the cost per share of trading $n$ shares is a fixed cost plus a cost proportional to the size of the trade. The permanent impact function is:

\[
\text{permanent impact} = \gamma_i \, \frac{n}{\tau}
\]

(2)

The utility function approach is used to establish that each point across the efficient frontier represents the unique optimal execution strategy for a trader with a certain degree of risk aversion. The latter’s optimal strategy in other words can be identified by appropriately minimizing the liquidation cost utility function.

The expected cost of trading stock $i$ along the trading trajectory is written $E[cost|N_i]$ and is given by the formula:

\[
E[cost|N_i] = \frac{1}{2} \gamma N_i^2 + c \sum_j |n_{ji}| + \frac{\eta}{\tau} \sum_j n_{ji}^2
\]

(3)

If $x_{ij}$ is the number of units of asset I held in the portfolio at time $j$ then the variance of liquidation cost along trajectory $N_i$ is given by the formula:

\[
\text{Var}[cost|N_i] = \sigma^2 \sum_j \tau x_{ji}^2
\]

(4)

The cost on $N$ can be defined as:

\[
C[cost|N] = E[cost|N] + \lambda \text{Var}[cost|N]
\]

(5)
The parameter $\lambda$ is the trading risk aversion. For a fixed level of expected cost and variance of cost increasing risk aversion increases cost. The idea of optimal liquidation is to find the trajectory that minimizes cost.

$$\min_{\mathcal{N}} E[\mathcal{N}] + \lambda V[\mathcal{N}]$$

The minimization takes place across all trading trajectories. The problem can be solved explicitly in the case where stocks follow an arithmetic random walk and impact functions are quadratic in total transaction size. For a portfolio consisting of a single stock the optimal trajectory is given by the formula:

$$n_j = \frac{\sinh(\kappa(T - t_j))}{\sinh(\kappa T)}N$$

$$\kappa \sim \sqrt{\frac{\lambda \sigma^2}{\eta} + O(\tau)}$$

With these formulas we can calculate the expected cost of liquidation and therefore the value under liquidation. The parameters we need are:

- The total time for liquidation $T-t_0$
- The risk aversion coefficient $\lambda$
- The temporary impact cost parameter $\eta$
- The permanent impact cost parameter $\gamma$
- The stock’s volatility $\sigma$
- The total number of units to liquidate $N$

Subramanian and Jarrow (2001) in their model tried to include, apart from the market impact of sales on stock price, the execution lag (where sales are not executed immediately after the order comes). Their model uses a Brownian motion for the price of stock. A price discount function which is non-increasing in sales is used for the impact of sales on price. The execution lag is expressed by a non-decreasing function of sales. If there is no liquidity risk, clearly the optimal trading strategy is block liquidation of stocks either at terminal date or immediately, depending on which way prices are moving. In conditions of liquidity risk, the best strategy is the same only if the total price discount incurred with the sale of two blocks of shares is less than or equal to the price discount incurred with the sale of all the shares in one go. In these two models an externally set fixed horizon for liquidation is required.

Trying to overcome this issue Hisata and Yamai (2000) introduced a model with continuous time approximation and assuming a constant speed of sales. In their model optimal execution strategy is derived in a closed-form solution taking into account the market impact of trader’s dealings using a mean and standard deviation approach. VaR is adjusted according to level of liquidity and the scale of the trader’s position. Certain assumptions are made and the sales completion time is treated as an endogenous variable. Their calculation is very practical and can be used even when
market impact is uncertain or has a non-linear relationship with sales volume. It is also suitable for multiple security portfolios.

Bangia, Diebold, Schuermann & Stroughair (1999) distinguished between exogenous liquidity factors which are more or less the same for all traders and endogenous liquidity factors which are different for every trader depending on the volume of the holding position, given that once the volume is greater than quote depth, the trading size starts to affect bid ask prices. This means that when there is not enough liquidity in the market the liquidation is not executed at the mid-price and this price has to be adjusted for the value of existing spread. To calculate usual VaR they consider the worst price of the stock for a certain confidence level and then consider the worst value of the spread to find the effect of the spread on the transaction price.

The next generation of models uses stochastic dynamic programming to derive the optimal trading strategy through maximization of the expected return of the trade. The main two approaches are those of Krokhmal and Uryasev (2003) and more recent non-parametric liquidity adjusted model proposed by Fragniere, Gondzio, Tuchschmid and Zhang in 2010.

Stochastic Programming has grown during recent years to deal with problems of decision making under conditions of uncertainty and is able to cope with optimization problems under uncertainty over time. In this framework the optimal execution strategy is highly dynamic taking into account the market conditions at every time interval. In order to model uncertainty using SP we generate future scenarios based on available historical data. Thus we get an approximation of future conditions because the liquidation process is multi-period problem. The evolution of stochastic parameters is often modelled using multinomial scenario trees where the branches generated by each node represent the uncertainty. The problem with this approach is that if we increase the number of branches, trying to better approximate uncertainty, the number of nodes increases exponentially leading to greater computational difficulty. To overcome this problem we can use Monte Carlo simulation to generate a group of sample paths to approximate future uncertainty. Each simulated path represents a future scenario. In this way an increase in the number of paths in an attempt to better capture uncertainty brings about a linear increase in the number of nodes even when the time period is increased.
Krokhmal and Uryasev (2003) developed a model for optimal trading strategy based on mathematical programming and sample path scenarios simulation. Some distinguishing characteristics of their model include the dynamic adaptation of the strategy to the market condition, the inclusion of various types of restrictions in the trading strategy (both institutional constraints and limitation deriving from trader risk preferences), incorporation of different models for calculating temporary and permanent market impact. Their simulations showed that when the temporary or permanent market impact is high, the optimal execution strategy is very close to the “naïve strategy”, meaning the sale of equal amounts of stocks at each time interval. When market impact is low, the optimal strategic choice begins to differentiate from the latter.

A more recent approach is the non-parametric liquidity adjusted model proposed by Fragniere, Gondzio, Tuschschmid and Zhang in 2010. Their model is an extension of Almgren and Chriss’s mean-variance approach. The optimal trading strategies arrived at by the latter do not react to the dynamics of the changing market situation because they are based on a “static” framework. For example, if there is an increasing trend observed in a stock price by other traders, they may make the decision to delay their own liquidation process. Conversely, if there is a market shock they may decide to speed up the liquidation. A closed form model does not respond dynamically to uncertainty of this type. To make this clear we will present their approach more analytically.

Let’s assume a collection of sample paths

\[ \mathcal{Z} = \left\{ \left( C_0, C_{1,z}, C_{2,z}, \ldots, C_{k,z}, \ldots, C_{N,z} \right) \mid s = 1, \ldots, Sc \right\} \]

where \( C_{k,s} \) represents information on relevant parameters. In the first stage we assume that randomness in the sample paths is provided only by the market price component \( \tilde{S}_k \). In the next stages, a random element can be extended to other parameters such as bid-ask spread and coefficients for temporary and permanent market impact. In this framework the trading strategy is represented by a two dimensional matrix rather than a vector.

\[
\text{strategy} = \begin{bmatrix}
  n_{1,1} & \cdots & n_{k,1} & \cdots & n_{N,1} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  n_{1,s} & n_{k,s} & \cdots & \cdots & n_{N,s} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  n_{1,Sc} & \cdots & n_{k,Sc} & \cdots & n_{N,Sc}
\end{bmatrix}
\]

First stage \hspace{1cm} Second stage
where $n_{k,s}$ is the quantity of shares sold in $k$th interval on path $s$, $s$ is the index of scenarios and $Sc$ is the number of generated scenarios. The first stage variables are locked. The actual sale price is reformulated from Almgren’s model:

$$
\hat{S}_{k,s} = \hat{S}_{k,s} - \frac{1}{2} \gamma \sum_{j=1}^{N} n_{j,s} - \frac{1}{2} \epsilon - \frac{\eta n_{s}}{\tau}
$$

The total amount realized from the sale under each scenario is the sum of the amounts realized across the whole set of $N$ intervals:

$$
total\,proceed = \sum_{k=1}^{N} n_{k,s} \hat{S}_{k,s} = \sum_{k=1}^{N} \left( \hat{S}_{k,s} n_{k,s} - \frac{1}{2} \gamma n_{k,s} - n_{k,s} \frac{1}{2} \gamma \sum_{j=1}^{N} n_{j,s} - \frac{\eta n_{s}^{2}}{\tau} \right)
$$

The liquidation cost of path $s$ is the difference between the above and the initial value of the portfolio:

$$
LC_{s} = XS_{0} - \sum_{k=1}^{N} n_{k,s} \hat{S}_{k,s} = XS_{0} + \frac{1}{2} \gamma X + \frac{1}{2} \gamma X^{2} - \sum_{k=1}^{N} \hat{S}_{k,s} n_{k,s} + \left( \frac{\eta X}{\tau} - \frac{1}{2} \gamma \right) \sum_{k=1}^{N} n_{k,s}^{2}
$$

The problem now transform to minimize the liquidation cost:

$$
\min_{n_{k,s}} \sum_{z=1}^{Sc} p_{s} \cdot LC_{s} = \sum_{s=1}^{Sc} p_{s} \cdot LC_{s}
$$

s.t.

$$
X = \sum_{k=1}^{N} n_{k,s} \quad \forall s = 1, ..., Sc
$$

$$
n_{1,s}, ..., n_{N,s} \geq 0 \quad \forall s = 1, ..., Sc
$$

$$
n_{1,s} = n_{1,s-1} \quad \forall s = 2, ..., Sc
$$

Where $p_{s}$ is the probability of scenario $s$ and is equal to $1/Sc$, since the scenarios generated by Monte Carlo have equal probabilities. The problem is then a quadratic optimization problem. From the solution of this optimization problem we derive the optimal trading strategy matrix. Then we apply this strategy to the corresponding scenario and arrive at the liquidation cost for this scenario, given by a vector indexed by $s$. Sort the vector $LC$ and find the value of the $a$th percentile $LC$. The most commonly used confidence level is 95 and 99.

**Empirical analysis**

After analyzing the concepts of VaR and LVaR we now proceed to the empirical analysis part. As made clear above, liquidity is the readiness with which an asset can be converted into cash. This readiness describes the degree of transaction costs, time and uncertainty that must be born in order to effect a transaction. Let’s consider a portfolio of stocks held by a financial institution. The value of this portfolio will depend in certain situations on the stocks’ liquidity. If the institution
needs to liquidate the portfolio to cover short term obligations that are coming due, the true value of the portfolio will depend on the expectation of turning it into cash over a specified period of time. We will examine this situation and find the value under liquidation to determine the true value of a portfolio when it has to be liquidated over a fixed time horizon.

Value under liquidation can be seen as an expected value with an associated risk. The finite liquidity of the market makes it impossible to instantly liquidate a portfolio and so the value under liquidation is random with a given mean and a standard deviation. As we cannot know for sure what value we will receive, the best we can do is determine the statistical average that liquidation will yield. We will calculate VaR and LVaR under different models and compare the results to find conclusions.

In determining a portfolio’s value under liquidation we must start by determining the degree of risk aversion. Risk aversion is an expression of the level of tolerance for discrepancies between the realized trading revenue and the actual trading revenue. So one can raise the value of a portfolio under liquidation simply by willing to take on more risk. Once the risk tolerance is determined there is a unique optimal trading trajectory associated with it. This trajectory is efficient, meaning that it has the minimal possible trading costs for its level of market exposure. So the value under liquidation of a portfolio is the expected cost of trading along an optimal trajectory for a given level of risk aversion.

Data

We assume that in our analysis the portfolios will consist of different stocks from the S&P 500 index, this being a widely accepted indicator of large-cap US equities and their risk/return characteristics. Over USD 7.8 trillion are benchmarked to the index and the index assets comprise approximately USD 2.2 trillion. The index includes 500 leading companies and covers approximately 80% of available market capitalization. The 500 stocks are chosen for market size, liquidity and industry grouping, among other factors and they are included in the index weighted by their respective market value. . .

For the needs of our analysis we downloaded historical data for all stocks of S&P 500 index in the form of continuous time series from 1991 until 2015. The source was Yahoo finance and the data include for every trading day open and close price, high and low price and volume of transactions.

Historical volatility measures

Various estimators have been developed to approach historical volatility, it. The calculation is affected by how many historical days are used for the calculation, as well as the estimation of the drift. Another tricky issue how long to go back in time. Given the fact that we want to capture the recent volatility in order to make future approximations we must not go back in time more than one year. The different estimators for the calculation of historical volatility variously use open(O), high(H), low(L) and close(C) daily prices. The most commonly used estimators are close-to-
close volatility, exponentially weighted volatility and some more complex measures (e.g., Parkinson, Garman-Klass, Rogers and Satchell & Yang-Zhang).

The most simplistic way to calculate volatility is to calculate first the daily log returns of the stock. Then to calculate the average log return and the standard deviation over the sampled period. With this approach problems may occur if we have a very high or negative return over the sampled period because in the long run such returns are unrealistic. Volatility calculation is more reliable if we assume zero return because we avoid corruption by taking for granted that past sample returns can reflect future returns. If then it is assumed that the return over the period is the same for all periods and that the mean return is zero, the standard deviation of the percentage change is the absolute value of the percentage return. To calculate daily volatility we divide annualized volatility by the square root of 250, which is the annual number of samples.

The parameters of length of time and frequency of measurement are important for calculating historical volatility. Many investors consider using the exact number of days of historical volatility observations as the implied duration of volatility of interest. This may indicate a realistic minimum and maximum value over a long period of time, but the reasonable level of long-dated implied volatility is not always best represented by the identical days of historical volatility given that average volatility reinstates itself over some 8 months. Periods which are a multiple of 3 months are suggested so that the historical volatility measure always includes an equivalent number of quarterly reporting dates. Any sudden increase in price in this period has to be excluded from the calculation since it is not expected to reoccur. If there is a unique economic occurrence which gave rise to a spike in volatility, future volatility may best be estimated by the past historical volatility when a similar event occurred. Concerning the frequency of measurement, daily or weekly data is usually used. We will use daily volatility as it provides five times more data points.

The liquidity estimator we are going to use in our analysis is the Garman-Klass because it is considerably more efficient than the close-to-close estimate and is the most effective for stocks characterized by Brownian motion, with no drift and no opening jumps. It was created as a development of the Parkinson measure and incorporates opening and closing prices. Overnight jumps are ignored meaning that volatility is underestimated. It is given by the following formula.

\[
\text{Volatility}_{\text{Garman-Klass}} = \sigma_{\text{GK}} = \sqrt{\frac{F}{N}} \sqrt{\sum_{i=1}^{N} \frac{1}{2} \left( \ln \left( \frac{h_i}{l_i} \right) \right)^2 - \left( 2\ln(2) - 1 \right) \left( \frac{\ln \left( \frac{C_i}{D_i} \right)}{C_i} \right)^2}
\]

**Bid-ask spread estimator**

With respect to the bid-ask spread, we are going to use here an estimator developed by Corwin and Schultz (2011) which estimates bid-ask spreads based on daily high and low prices. The authors start with two basic observations: First, that daily high prices are generally transactions initiated by buyers whereas daily low prices are generally trades initiated by sellers. The daily ratio of high-to-low prices

-19-
consequently has two components reflecting the fundamental volatility of the stock and its bid-ask spread. The second observation is that the volatility component increases in proportion with the trading interval but the bid-ask spread component does not. As a result according to the authors “the sum of the price ranges over two consecutive single days reflects two days’ volatility and twice the spread, while the price range over one two-day period reflects two days’ volatility and one spread”. Consequently a stock’s bid-ask spread estimate can be calculated from the high-to-low price ratio for a two-day period and the high-to-low ratios for two consecutive single days.

Back testing simulations included in their paper show that under normal circumstances, high-low spread estimates correlate with true spreads with a coefficient of about 0.9 and the standard deviation is much smaller (by fifty to seventy-five percent) than the standard deviation of estimates from other covariance estimators and specifically the Roll (1984) covariance estimator. The reason we chose this estimator in our analysis is because it has some advantages compared with the existing measures. Apart from the fact that it has better results than the Roll measure, it is also very easy to use. In addition, unlike Gibbs or Holden estimators, it is not computer-time sensitive and is therefore better for large samples.

For stocks, such as those listed on S&P 500, the high-low spread also outperforms other estimators in the way it captures the time-series variation in individual stock spreads and gives a broader picture of liquidity than simple bid-ask spread. There are cases where trades are executed at daily high or low prices, for example when the market impact from large orders affects the price or in the case of a series of buy or sell orders taking place in a shallow market. The high-low spread estimator is able to capture such temporary price effects as well as the bid-ask spread. It can be calculated from the following formulas:

\[
\beta = E \left\{ \sum_{j=0}^{1} \left[ \ln \left( \frac{H_{t+j}}{L_{t+j}} \right) \right]^2 \right\}
\]

\[
\gamma = \left[ \ln \left( \frac{H_{t,t+1}}{L_{t,t+1}} \right) \right]^2.
\]

\[
\alpha = \frac{\sqrt{2\beta} - \sqrt{\beta}}{3 - 2\sqrt{2}} - \frac{\gamma}{3 - 2\sqrt{2}}.
\]

\[
S = \frac{2(e^\alpha - 1)}{1 + e^\alpha}.
\]
Numerical calculations for a single stock portfolio

In this section we examine a portfolio consisting of Morgan Stanley stocks of $1,000,000. The historical data selected for the period from April 2001 to April 2002. Portfolio VaR was calculated with three methods (historical, Delta-normal, Monte Carlo) for a liquidation period of 10 days with a time interval of 1 day.

For the historical simulation we plotted the daily log returns for the 250 days of observations and a histogram of the returns. Using data from more than one year is not suggested for historical calculation method. Most daily returns are clustered near 0.

Var is now a percentile function of our confidence level. The results are shown below for confidence levels 95% and 99%

<table>
<thead>
<tr>
<th></th>
<th>VaR 95%</th>
<th>VaR 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily</td>
<td>4.69%</td>
<td>$46,900</td>
</tr>
<tr>
<td>Daily</td>
<td>6.13%</td>
<td>$61,300</td>
</tr>
<tr>
<td>10 day</td>
<td>14.84%</td>
<td>$148,400</td>
</tr>
<tr>
<td>10 day</td>
<td>19.37%</td>
<td>$193,700</td>
</tr>
</tbody>
</table>
With the delta-normal method, we assume a normal distribution of the log returns and its mean and standard deviations. For the mean we use the average daily return for the last year which is calculated -0.11%. For the standard deviation (volatility) we use the Karman-Glass volatility estimator and the annualized volatility calculated 40.48%. For the daily volatility we divide with the square root of 250. The calculated value is 2.56%. VaR is now given from the formula $\text{VaR} = \mu \cdot T + \alpha \cdot \sigma \cdot \sqrt{T}$. The results are shown below for confidence levels 95% and 99%.

<table>
<thead>
<tr>
<th></th>
<th>VaR 95%</th>
<th>VaR 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily VaR</td>
<td>4.1%</td>
<td>$41,000</td>
</tr>
<tr>
<td>Daily VaR</td>
<td>5.84%</td>
<td>$58,400</td>
</tr>
<tr>
<td>10 day VaR</td>
<td>12.21%</td>
<td>$148,361</td>
</tr>
<tr>
<td>10 day VaR</td>
<td>17.73%</td>
<td>$177,300</td>
</tr>
</tbody>
</table>

To estimate VaR using Monte Carlo we simulated 10,000 scenarios for a ten day horizon. The geometric Brownian motion model were used to simulate prices. Returns on corresponding initial prices were then calculated using the simulated prices. Portfolio returns at the 1% and 5% points on the full scale of returns were then identified, estimating VaR at the corresponding confidence levels of 99% and 95%. The simulated scenarios and the results are shown below.

<table>
<thead>
<tr>
<th></th>
<th>VaR 95%</th>
<th>VaR 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 day VaR</td>
<td>9.41%</td>
<td>$941,000</td>
</tr>
<tr>
<td>10 day VaR</td>
<td>11.78%</td>
<td>$117,800</td>
</tr>
</tbody>
</table>
The next step is to calculate the liquidity adjusted VaR which can be expressed as \[ \text{LVaR} = \text{VaR} + \text{LC} \]

To estimate Liquidation Cost we need to calculate the daily bid-ask spread for a year of historic prices. The daily bid-ask spread was calculated using R with the method and the formulas described above. In the first stage liquidation cost was calculated as half of the worst expected spread. The biggest the spread the less liquid is the stock and greater the VaR must be. We use the half of the spread in the calculations because the whole spread represents the cost of immediacy for a round trip (buy/sell) and we only want to sell.
Two different calculations were made for two different estimated values for the worst spread. One with historic estimation as a percentile function of the 250 calculations of the spread across one year with a confidence level of 95%. The second by calculating the mean and the standard deviation of the spread and then calculating the worst expected spread. We can see that from historic spreads we get higher values of LC. If the liquidation take place during crisis the worst spread will be more close to its historic highs rather than close to last year’s higher deviation.

<table>
<thead>
<tr>
<th>Method of estimation</th>
<th>Worst expected spread 95%</th>
<th>Worst expected spread 99%</th>
<th>LC 95%</th>
<th>LC 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Historic</td>
<td>7.13%</td>
<td>8.51%</td>
<td>3.57%</td>
<td>4.26%</td>
</tr>
<tr>
<td>m=3.8% , σ=1.56%</td>
<td>6.37%</td>
<td>7.44%</td>
<td>3.19%</td>
<td>3.72%</td>
</tr>
</tbody>
</table>

To examine the effects of both endogenous and exogenous liquidity factors we will describe LVaR in the context of finding an optimal execution strategy and the efficient frontier described in the Almgren & Chriss model. The stock price is affected by two exogenous factors and one endogenous factor, market impact. The exogenous factors, volatility and drift, are brought about by market forces and are not affected by the transactions carried out by the trader. When the market starts to feel the effect of the volume being sold, the bids will adjust downward. Market impact takes two forms: temporary impact is from supply and demand differences arising from the trader’s transactions causing temporary price changes away from the equilibrium price; permanent impact refers to the equilibrium price changing as a result of the transactions of the trader, this change holding for at least the period of the liquidation. The calculation of optimal trajectories becomes simpler if we assume that market impact functions are linear. (Formulas 1-2)

To find the optimal trajectory means to find the trajectory that minimizes the equation (5). E and V of the linear impact functions is calculated for the two most extreme trajectories, which are selling at a constant rate at each time interval or selling to minimize variance regardless of the transaction costs. The most straightforward trajectory is that of selling at a constant rate along the entire time horizon. This trajectory minimizes total expected costs on the one hand but the variance may become considerable the longer the horizon. Block liquidation of the whole position straightaway lies at the other extreme. The corresponding trajectory has zero variance due to how time is discretized in this model. If N is large and τ is therefore short, the entire starting position may incur an exceptionally large price penalty.

The optimal trajectory is somewhere between and is given by equation (6) for small time step τ. On the condition that X>0, n_j>0 for each j. When selling a large initial long position, therefore, the solution according to Almgren and Chriss (2000) “decreases monotonically from its initial value to zero at a rate determined by parameter κ”. With short trading intervals, κ^2 expresses the ratio of volatility multiplied by the relevant risk intolerance to the parameter of temporary transaction
cost. The formulas below show the calculation of expectation and variance of the optimal strategy for a given portfolio size $X$:

$$E(X) = \frac{1}{2} \gamma X^2 + \epsilon X + \eta X^2 \frac{\tanh\left(\frac{1}{2} \kappa T\right) \left(\tau \sinh(2\kappa T) + 2T \sinh(\kappa T)\right)}{2\tau^2 \sinh^2(\kappa T)}$$

$$V(X) = \frac{1}{2} \sigma^2 X^2 \frac{\tau \sinh(\kappa T) \cosh(\kappa (T - \tau)) - \tau \sinh(\kappa T)}{\sinh^2(\kappa T) \sinh(\kappa T)}$$

In the following graph we can see the efficient frontier of our portfolio. Each point on the frontier represents a single strategy of optimal liquidation of our portfolio. The shaded part is the set of variances and expected losses that may attain from various time-dependent strategies. The efficient frontier is represented by the solid curve while strategies with higher variance for the same expected cost lie on the dotted line. Point B, i.e., the point of lowest expected cost on the curve, represents the naïve strategy of minimizing expected losses regardless of the variance. A straight line tangent to the curve represents a linear optimal strategy, such as the one shown, for $\lambda=10^{-6}$.

The strategy trajectories for points A, B and C are given in the following graph, starting from a holding of $1,000,000$ (~21,000 shares) on day one and liquidating during the 10-day horizon. Trajectory A has $\lambda=2\times10^{-6}$ and reflects the risk-averse trader focusing on selling quickly to avoid volatility risk, even though he incurs a high trading cost by doing so. Trajectory B has $\lambda=0$ and reflects the naïve strategy mentioned above. Given zero drift and linear transaction costs, this is effectively a straightforward regular reduction of the holding throughout the time horizon of the liquidation. This will never be an optimal strategy because variance can be reduced considerably in return for a relatively inconsequential transaction cost increase. Trajectory C has $\lambda=2\times10^{-6}$ and reflects a trader who likes risk. By postponing the sells, he can expect higher trading costs, explained by the speed of
sales at the end, and also higher variance over the longer time horizon for which he keeps the stock.

By incorporating market impact to our calculation the value of the LVaR depends also on the initial holding. In order to illustrate how the calculation of LVaR is affected by initial position, LVaR was estimated with five different initial holdings. Market impact parameters $\gamma$, $\eta$ are calculated taking into account the average of the daily trading volume, which is 4.8 million shares. With respect to $\eta$ every percent of the daily volume traded is assumed to have an effect on price corresponding to the bid-ask spread. The value of the rest of the parameters was chosen as shown below:

- Initial stock price $S_0 = 47.7$$/share
- Liquidation time $T = 10$ days
- Number of time periods = 10
- Annual volatility = 40.48%
  \[ \sigma = 1.22 \text{ ($/share)day}^{1/2} \]
- Annual growth = -0.11%
  \[ \alpha = 21 \times 10^{-5} \text{ ($/share)/day} \]
- Bid-ask spread $\epsilon = 3.04$ $$/share

<table>
<thead>
<tr>
<th>Initial holding (stocks)</th>
<th>1,000,000</th>
<th>500,000</th>
<th>100,000</th>
<th>50,000</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>LVaR 95%</td>
<td>2,108,000</td>
<td>878,000</td>
<td>141,100</td>
<td>68,900</td>
<td>13,540</td>
</tr>
<tr>
<td>LVaR per share</td>
<td>2.108</td>
<td>1.756</td>
<td>1.411</td>
<td>1.378</td>
<td>1.354</td>
</tr>
<tr>
<td>LVaR ratio</td>
<td>4.42%</td>
<td>3.68%</td>
<td>2.96%</td>
<td>2.89%</td>
<td>2.84%</td>
</tr>
</tbody>
</table>
**Numerical calculations for multiple stocks portfolio**

In this section we examine a portfolio consisting of three stocks Morgan Stanley, JP Morgan and Bank of America. The initial value of the portfolio is $3,000,000 and is equally distributed to the three stocks. The historical data selected for the period from April 2001 to April 2002. The liquidation horizon is chosen to be 10 days and the time interval 1 day.

For the historical simulation we collected the daily prices for 250 days and calculated the daily log returns of each stock. Then we created a time series of the daily returns/losses in $ for each stock. By summing them up we created a time series of daily portfolio returns/losses in $. This is plotted below along with a histogram of the return.
Var can now be calculated as a percentile function of the daily returns. The calculation are shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Confidence 95%</th>
<th>Confidence 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Daily VaR in $</td>
<td>$ -92.687</td>
<td>$ -175.161</td>
</tr>
<tr>
<td>Daily VaR</td>
<td>3.09%</td>
<td>5.84%</td>
</tr>
<tr>
<td>10 day VaR in $</td>
<td>$ -293.102</td>
<td>$ -553.909</td>
</tr>
<tr>
<td>10 Day VaR</td>
<td>9.77%</td>
<td>18.46%</td>
</tr>
</tbody>
</table>

The second method used to calculate VaR is the variance-covariance matrix. First we calculated the daily log returns of the three stocks for each day for 250 days. Then we calculate the mean, variance and standard deviation of the returns of each stock. The results are shown below.

<table>
<thead>
<tr>
<th>Morgan Stanley</th>
<th>Bank of America</th>
<th>JP Morgan</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.11%</td>
<td>0.10%</td>
<td>-0.13%</td>
</tr>
<tr>
<td>0.00086</td>
<td>0.000314</td>
<td>0.000521</td>
</tr>
<tr>
<td>2.94%</td>
<td>1.77%</td>
<td>2.29%</td>
</tr>
</tbody>
</table>

Then we calculated the matrix of excess returns by subtracting from the daily returns the average returns. The variance-covariance matrix is formed by multiplying the matrix of excess returns with itself transposed and dividing by the number of observation. The variance-covariance matrix is shown below:

<table>
<thead>
<tr>
<th></th>
<th>Morgan Stanley</th>
<th>Bank of America</th>
<th>JP Morgan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morgan Stanley</td>
<td>0.000860122</td>
<td>0.000264056</td>
<td>0.000448</td>
</tr>
<tr>
<td>Bank of america</td>
<td>0.000264056</td>
<td>0.000313638</td>
<td>0.000246</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>0.000448284</td>
<td>0.000245571</td>
<td>0.000521</td>
</tr>
</tbody>
</table>

In the next step we calculate the mean return of the portfolio by multiplying the matrix of mean returns of each stock with a matrix of portfolio weights. To calculate the portfolio sigma we multiply the transposed matrix of the portfolio weights with the variance-covariance matrix and the result we multiply again with the portfolio weights matrix. So we calculate the variance of the portfolio. Sigma is the square root of the variance. Knowing the sigma and mean we calculate the daily and 10 day VaR for confidence levels of 95% & 99%. The results are shown below.
<table>
<thead>
<tr>
<th></th>
<th>$</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>investment</td>
<td>$ 3,000,000</td>
<td></td>
</tr>
<tr>
<td>Mean Return</td>
<td>$ 1.327</td>
<td>-0.044%</td>
</tr>
<tr>
<td>Portfolio sigma</td>
<td>$ 60,086</td>
<td>2.003%</td>
</tr>
<tr>
<td>Mean investment</td>
<td>$ 2,998,673</td>
<td></td>
</tr>
<tr>
<td>Sigma of investment</td>
<td>$ 60,059</td>
<td></td>
</tr>
<tr>
<td>Cut off 95%</td>
<td>$ 2,899.841</td>
<td></td>
</tr>
<tr>
<td>Daily VaR 95%</td>
<td>$ 100,159</td>
<td>3.34%</td>
</tr>
<tr>
<td>10 day Var 95%</td>
<td>$ 316,731</td>
<td>10.56%</td>
</tr>
<tr>
<td>Cut off 99%</td>
<td>$ 2,858,893</td>
<td></td>
</tr>
<tr>
<td>Daily VaR 99%</td>
<td>$ 141,107</td>
<td>4.70%</td>
</tr>
<tr>
<td>10 day Var 99%</td>
<td>$ 446,220</td>
<td>14.87%</td>
</tr>
</tbody>
</table>

Two different calculations were made for two different estimated values for the worst spread. One with historic estimation of each stock as a percentile function of the 250 calculations of the spread across one year with a confidence level of 95% and 99%. The second by calculating the mean and the standard deviation of the spread of each stock and then calculating the worst expected spread. Then by multiplying with the portfolio weights we can calculate the worst expected spread of the portfolio. The results are presented below.

<table>
<thead>
<tr>
<th>Historic</th>
<th>Worst expected spread 95%</th>
<th>Worst expected spread 99%</th>
<th>LC 95%</th>
<th>LC 99%</th>
<th>LC 95%</th>
<th>LC 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morgan Stanley</td>
<td>7.13%</td>
<td>8.51%</td>
<td>3.57%</td>
<td>4.26%</td>
<td>$35,700</td>
<td>$42,600</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>5.92%</td>
<td>8.68%</td>
<td>2.96%</td>
<td>4.94%</td>
<td>$29,600</td>
<td>$49,400</td>
</tr>
<tr>
<td>Bank of America</td>
<td>3.99%</td>
<td>5.83%</td>
<td>2%</td>
<td>2.91%</td>
<td>$20,000</td>
<td>$29,100</td>
</tr>
<tr>
<td>Portfolio</td>
<td>5.68%</td>
<td>7.67%</td>
<td>2.84%</td>
<td>3.84%</td>
<td>$85,300</td>
<td>$121,100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Delta Normal with μ &amp; σ</th>
<th>Worst expected spread 95%</th>
<th>Worst expected spread 99%</th>
<th>LC 95%</th>
<th>LC 99%</th>
<th>LC 95%</th>
<th>LC 99%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Morgan Stanley</td>
<td>6.37%</td>
<td>7.44%</td>
<td>3.19%</td>
<td>3.72%</td>
<td>$31,900</td>
<td>$37,200</td>
</tr>
<tr>
<td>JP Morgan</td>
<td>5.66%</td>
<td>6.68%</td>
<td>2.83%</td>
<td>3.34%</td>
<td>$28,300</td>
<td>$33,400</td>
</tr>
<tr>
<td>Bank of America</td>
<td>3.97%</td>
<td>4.67%</td>
<td>1.99%</td>
<td>2.34%</td>
<td>$19,900</td>
<td>$23,400</td>
</tr>
<tr>
<td>Portfolio</td>
<td>5.33%</td>
<td>6.26%</td>
<td>2.67%</td>
<td>3.13%</td>
<td>$80,100</td>
<td>$94,000</td>
</tr>
</tbody>
</table>
We can see that from historic spreads we get higher values of LC. If the liquidation take place during crisis the worst spread will be more close to its historic highs rather than close to last year’s higher deviation.

In the next stage we implement the Almgren and Chriss model for multiple asset portfolio in order to identify optimal strategies and the efficient frontier. With three securities, our position at each time interval k is a column vector $X_k = (X_{1k}, X_{2k}, X_{3k})^T$ (where $^T$ denotes transpose). $X_0 = X = (X_1, X_2, X_3)^T$ indicates starting value and the the column vector $n_k = x_{k-1} - x_k$ represents the trade lists. The column vector of stock prices $S_k$ is assumed to follow a multidimensional arithmetic Brownian random walk with zero drift, the dynamics of which can be expressed as in the case of a single stock but in this case $\xi_k = (\xi_{1k}, \xi_{2k}, \xi_{3k})^T$ is a vector of 3 independent Brownian increments. $C = \sigma^*\sigma^T$ is the 3x3 symmetric positive definite variance-covariance matrix. The permanent and temporary impact are vector functions of a vector. Considering the linear model, it can be expressed as follows, with $\Gamma$ and $\Pi$ 3X3 matrices and $\epsilon$ a 3X1 column:

$$g(v) = \Gamma v, \quad h(v) = \epsilon \text{sgn}(v) + \Pi v,$$

Despite the multidimensionality of the problem, two scalar functions give the set of all outcomes. The utility and VaR functions continue to be expressed in terms of E and V. The portfolio’s optimal liquidation strategy is again determined as a linear problem.

$$\frac{x_{k-1} - 2x_k + x_{k+1}}{\tau^2} = \lambda\Pi^{-1}C x_k + \Pi^{-1}\Gamma^A \frac{x_{k-1} - x_{k+1}}{2\tau},$$

A diagonal assumption is now made, meaning that the transactions in any one stock are taken to affect only the price of this stock and not the other prices. As a result $\Gamma$ and $\Pi$ matrices have to be diagonal. This assumption means that the number of coefficients needed correlates with the number of stocks. These coefficients can be calculated using the data we have available. Given the diagonal assumption, the solution will be a combination of exponentials $\exp(\pm \kappa_j t)$.

$$2 \frac{\cosh(\kappa_j \tau) - 1}{\tau^2} = \tilde{\kappa}_j^2,$$

$$z_{jk} = \frac{\sinh(\kappa_j(T - t_k))}{\sinh(\kappa_j T)} z_{j0},$$

$$z_0 = U^\top y_0 = U^\top \tilde{\Pi}^{1/2} X, \quad x_k = \tilde{\Pi}^{-1/2} U z_k.$$
In our case, as mentioned above, our portfolio consists of three bank stocks ($3,000,000 equally weighted) which are highly liquid and with relative high correlation of their price fluctuation. The initial parameters are shown below.

Initial stock price $S_0 = 47.7, 35.1, 72.5 \$/share  
Liquidation time $T = 10$ days  
Number of time periods = 10  
Annual volatility = 40.48\%_36.14\%_24.82\%  
Annual growth = -0.11\%_0.13\%_0.1\%  
Average daily volume = 4.8, 8.9, 11.8 million shares

In the following graph, we can see the efficient frontier for our portfolio which seems very similar to the case of one stock. The straight line represents the optimal strategy for $\lambda=10^{-6}$. Greater correlation of the two securities will increase the relationship between their trajectories. If the diagonal assumption is no longer made, the same effect, of greater interdependence, would be expected.

In order to illustrate how the position to start with affects the calculation of LVaR, the latter was estimated with five different initial holdings. The calculations are shown below

<table>
<thead>
<tr>
<th>Initial holding ($)</th>
<th>50,000,000</th>
<th>20,000,000</th>
<th>10,000,000</th>
<th>3,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>LVaR 95%</td>
<td>1,967,000</td>
<td>693,100</td>
<td>318,900</td>
<td>94,640</td>
</tr>
</tbody>
</table>

![Graph showing efficient frontier for portfolio]
Conclusions

Liquidity plays an important role in the identification and measurement of market risk but its use in a Value-at-Risk framework is still being researched. The realization of its potential significance followed a number of liquidity crises taking place over the last few decades which were not signaled by existing market risk models. Various ways of incorporating endogenous and exogenous liquidity risk into the VaR calculations were presented, giving a clear picture of the ways this can be achieved. One such is the use of the bid-ask spread, which is effective in measuring and following liquidity changes. It is also a measure which is easy to calculate using readily available real-time data. By introducing the temporary and permanent market impact and the choice of optimal liquidation strategy with respect to a portfolio, the calculation of liquidity-adjusted VaR can be seen to depend both on the quantity of the holding stocks and the liquidation strategy that the trader will follow along a specified liquidation time horizon.

The last crisis has shown that even an apparently healthily liquid market can suddenly face a liquidity crisis of such depth as to create serious and systemic failures in financial and wider markets. So market participants and policy makers need to set up policies in advance that will maintain market functioning during periods of crisis. Structural changes such as reductions in market making seem to have reduced the level of market liquidity. Other changes in market structures (such as higher concentration of holdings by mutual funds) appear to have increased the fragility of liquidity. On the other hand standardization and enhanced transparency appear to improve liquidity levels. This especially is an area in which regulatory measures have a significant role to play.
Bibliography


Discussion Paper on Defining Liquid Assets in the LCR under the draft CRR 2013


